Problems

1. (*a*) Show that the change of variables specified

$$\mathbf{q} = \frac{\partial \hat{S}(\mathbf{p}, \bar{\mathbf{q}}, t)}{\partial \mathbf{p}}, \quad \bar{\mathbf{p}} = \frac{\partial \hat{S}(\mathbf{p}, \bar{\mathbf{q}}, t)}{\partial \bar{\mathbf{q}}}$$

is symplectic.

(*b*) Find a function $\hat{S}(p_n, \theta_{n+1})$ in terms of which the map (7.15) is given by $\theta_n = \partial \hat{S}/\partial p_n$, $p_{n+1} = \partial \hat{S}/\partial \theta_{n+1}$.

2. Consider the following four-dimensional map (Ding *et al.*, 1990a),

$$\begin{aligned} x_{n+1} &= 2\alpha x_n - p_{x,n} - \rho x_n^2 + y_n^2, \\ p_{x,n+1} &= x_n, \\ y_{n+1} &= 2\beta y_n - p_{y,n} + 2x_n y_n, \\ p_{y,n+1} &= y_n. \end{aligned}$$

Is it volume preserving? Using Eq. (7.13) test to see whether the map is symplectic.

3. Consider a magnetic field in a plasma given by

$$\mathbf{B}(x, y, z) = B_0 \mathbf{z}_0 + \nabla \times \mathbf{A},$$

where B_0 is a constant and the vector potential $\mathbf{A}$ is purely in the z-direction, $\mathbf{A} = A(x, y, z)\mathbf{z}_0$. Denote the path followed by a field line as $\mathbf{x}(z) = x(z)\mathbf{x}_0 + y(z)\mathbf{y}_0 + z\mathbf{z}_0$. Show that the equations for $x(z)$ and $y(z)$ are in the form of Hamiltonian's equations where z plays the role of time and $A(x, y, z)/B_0$ plays the role of the Hamiltonian.

4. Consider the motion of a charged particle in an electrostatic wave field in which the electric field is given by $\mathbf{E}(x, t) = E_x(x, t)\mathbf{x}_0$ with

$$E_x(x, t) = \sum_{\kappa,\omega} E_{\kappa,\omega} \exp(i\kappa x - i\omega t).$$

(This situation arises in plasma physics where the wave field E_x is due to collective oscillations of the plasma.) In the special case where there is only one wavenumber, $\kappa = \pm k_0$, the frequencies ω form a discrete set, $\omega = 2\pi n/T$ (where T is the fundamental period and n is an integer; $n = \ldots, -2, -1, 0, 1, 2, \ldots$), and the amplitudes $E_{\kappa,\omega}$ are real and independent of ω and κ, $E_{\kappa,\omega} = E_0/2$, the above expression for E_x reduces to

$$\begin{aligned} E_x(x, t) &= E_0 \cos(k_0 x) \sum_n \exp(2\pi i n t/T) \\ &= E_0 \cos(k_0 x) \sum_m \delta(t - mT). \end{aligned}$$

Show that the motion of a charged particle is described by a map which is of the same form as the standard map Eq. (7.15).

5. Find the fixed points of the standard map (7.15) that lie in the strip $\pi > p > -\pi$. Determine their stability as a function of K. In what range of K is there an elliptic fixed point (assume $K \geq 0$)?

6. Write a computer program to iterate the standard map, Eqs. (7.15).

   (*a*) Plot p modulo 2π versus θ for orbits with $K = 1$ and the following five initial conditions, $(\theta_0, p_0) = (\pi, \pi/5), (\pi, 4\pi/5), (\pi, 6\pi/5), (\pi, 8\pi/5), (\pi, 2\pi)$.

   (*b*) For $K = 21$ plot versus iterate number the average value of p^2 averaged over 100 different initial conditions, $(\theta_0, p_0) = (2n\pi/11, 2m\pi/11)$ for $n = 1, 2, \ldots, 10$ and $m = 1, 2, \ldots, 10$, and hence estimate the diffusion coefficient D. How well does your numerical result agree with the quasilinear value Eq. (7.42)?

7. The 'sawtooth map' is obtained from the standard map, Eqs. (7.15), by replacing the function, $\sin\theta_{n+1}$ in (7.15b) by the sawtooth function, saw θ_{n+1}, where

$$\text{saw}\,\theta \equiv \begin{cases} \theta, & \text{for } 0 \leqslant \theta < \pi, \\ \theta - \pi, & \text{for } \pi < \theta \leqslant 2\pi, \end{cases}$$

   and saw $\theta \equiv$ saw $(\theta + 2\pi)$. Show that the sawtooth map is an example of a C-system if $K > 0$ or $K < -2$ and calculate the Lyapunov exponents.

8. A two-degree-of-freedom system has the following Hamiltonian in action-angle variables, $H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \epsilon V(\theta_1, \theta_2)$ where

$$H_0(J_1, J_2) = \Lambda J_1^{3/2} + \Omega J_2,\ V(\theta_1, \theta_2) = \cos\theta_1 \sum_{n=-\infty}^{\infty} V_n \exp(in\theta_2),$$

   Λ and Ω are constants, and ϵ is small.

   (*a*) Obtain an expression for the trajectory $J_1(t)$ to first order in ϵ.

   (*b*) Which tori in the phase space are destroyed by the perturbation?

   (*c*) What does the KAM theorem tell us about the phase space for small ϵ? Answer in several complete sentences.

Notes

1. Additional useful material on chaos in Hamiltonian systems can be found in the texts by Sagdeev *et al.* (1990), by Ozorio de Almeida (1988), by Lichtenberg and Lierberman (1983) and by Arnold and Avez (1968), in the review articles by Berry (1978), by Chirikov (1979) and by Helleman (1980), and in the reprint selection edited by MacKay and Meiss (1987).
2. See books which cover the basic formulation and analysis of Hamiltonian mechanics, such as Ozorio de Almedia (1988) and Arnold (1978, 1982).
3. Our review in Section 7.1 is meant to refresh the memory, rather than to be a self-contained first-principles exposition. Thus, the reader who wishes more detail or clarification should refer to one of the texts cited above.[2]
4. If only k-independent relations of the form $\mathbf{m} \cdot \boldsymbol{\omega} = 0$ hold with $1 < k < N - 1$, then orbits on the N-torus are $(N - k)$-frequency quasiperiodic and do not fill the N-torus. Rather individual orbits fill $(N - k)$-tori which lie in the N-torus.
5. In the area preserving case, the areas of lobes bounded by stable and unstable

manifold segments must be the same if these lobes map to each other under iteration of the map. For example, consider one of the finger shaped areas bounded by stable and unstable manifold segments in Figure 4.10(c). This area must be the same as the areas of the regions shown in the figure to which it successively maps.

6. Long-time power law correlations of orbits have been observed numerically in two-dimensional maps (Karney, 1983; Chirikov and Shepelyanski, 1984) and in higher-dimensional systems (Ding *et al.*, 1991a). This comes about due to the 'stickiness' of KAM surfaces: an orbit in a chaotic component which comes near a KAM surface bounding that component tends to spend a longer time there, and this time is typically longer the nearer the orbit comes. This behavior has been examined theoretically using self-similar random walk models (Hanson *et al.*, 1985; Meiss and Ott, 1985). This type of behavior has also been shown to result in anomalous diffusion wherein the average of the square of a map variable increases as n^α with $\alpha > 1$ (in contrast to ordinary diffusive behavior where $\alpha = 1$ as in Eq. (7.44)). See Geisel *et al.* (1990), Zaslavski *et al.* (1989) and Ishizaki *et al.* (1991).

7. Note that near the peaks of the graph in Figure 7.17 it appears that the numerically computed D values can be much larger than the analytical estimate. In fact, it was subsequently found that D diverges to infinity in these regions, and the actual behavior of $\langle p^2/2 \rangle$ is anomalous[6] in that $\langle p^2/2 \rangle \sim n^\alpha$ with $\alpha > 1$. This behavior is due to the presence of 'accelerator modes' in the range of K-values near the peaks of the graph. Accelerator modes are small KAM island chains such that, when an orbit originates in an island, it returns periodically to that island but is displaced in p by an integer multiple of 2π. Hence, the orbit experiences a free acceleration, $p \sim n$. Orbits in the large chaotic region can stick close to the outer bounding KAM surfaces of these accelerator islands, thus leading to the above mentioned anomalous behavior[6] (Ishizaki *et al.*, 1991).

8. We note, however, that, if we consider the $\tilde{q}$ times iterated map, then there may be mixing regions in σ for the map $\mathbf{M}^{\tilde{q}}$, since $\mathbf{M}^{\tilde{q}}(\sigma) = \sigma$.